Department of Applied Mathematics, University of Venice

WORKING PAPER SERIES

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Working Paper n. 144/2006
November 2006

ISSN: 1828-6887

Incomplete pairwise comparison and consistency optimization

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November, 2006

The final version is forthcoming in the *European Journal of Operation Research, 2006*

Abstract. This paper proposes a new method for calculating the missing elements of an incomplete matrix of pairwise comparison values for a decision problem. The matrix is completed by minimizing a measure of global inconsistency, thus obtaining a matrix which is optimal from the point of view of consistency with respect to the available judgements. The optimal values are obtained by solving a linear system and unicity of the solution is proved under general assumptions. Some other methods proposed in the literature are discussed and a numerical example is presented.

Keywords: consistency, pairwise comparison matrices.

JEL Classification Numbers: C6.

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1 Introduction

Pairwise comparison is a well established technique in decision making. In T. Saaty's AHP [17], as an example, pairwise comparison matrices (*PCM* in the following) are used to derive the priorities for n alternatives by means of the so–called eigenvector method.

Nevertheless, in some cases we have to face a problem with missing judgements, thus obtaining an incomplete comparison matrix. This may happen, for instance, when the number of the alternatives, n, is large. In such cases it may be practically impossible, or at least unacceptable from the point of view of the expert, to perform all the $n(n-1)/2$ required comparisons to complete the *PCM*. A trade-off between the completeness of the available information and the need to keep reasonably small the number of questions to be submitted to the expert is then unavoidable. Moreover, as it has been pointed out in [10], it can be convenient/necessary to skip some direct critical comparison between alternatives, even if the total number of alternatives is not large. Some methods have been proposed in the literature to derive the priorities of n alternatives from an incomplete $n \times n$ *PCM* [1] [4] [10] [11] [15] [19] [23] [24].

In this paper we define a measure of the inconsistency of a *PCM* using an index introduced in [7] and then we propose to calculate the missing elements of an incomplete *PCM* by maximizing the global consistency (i.e. minimizing the inconsistency) of the 'completed' matrix. The paper is organized as follows: In Section 2 we define the problem, we introduce the necessary notations and briefly describe some methods proposed in the literature to solve the problem of incomplete comparisons. In Section 3 we describe our method and we solve the one and two–dimensional cases (i.e. when one or two comparisons between the alternatives are missing); then we extend our results for the general case of p missing comparisons. Finally, we present some results obtained by applying our method on a case proposed in [18] by T. Saaty.

2 Incomplete comparisons

2.1 The problem formulation

Let $\Lambda = \{A_1, A_2, ..., A_n\}$ be a set of n alternatives and let the judgements of a decision maker be expressed by pairwise comparisons. If all pairs of alternatives (A_i, A_j) with $i < j$ are considered, then it is necessary to perform

$n(n-1)/2$ comparisons. With this data, one can obtain the upper diagonal triangle of an $n \times n$ matrix. The remaining elements of the matrix are easily derived, as it is usually assumed that by comparing A_i with A_j, the comparison of A_j with A_i is automatically assigned. Clearly, the elements of the diagonal need not to be computed.

There are two main approaches to the problem of quantifying the comparative judgements. The first is the Saaty's approach, which is also called *multiplicative*. In this framework, a_{ij} estimates the relative preference of the alternative A_i when it is compared with the alternative A_j. The obtained *multiplicative PCM* $A = [a_{ij}]$ is positive, reciprocal ($a_{ji} = 1/a_{ij}$), with $a_{ii} = 1,\ i = 1, ..., n$ [16] [17]. To estimate the comparison a_{ij}, Saaty proposes to use the rational numbers $1/9, 1/8, ..., 1, 2, ..., 9$ with the following meaning: if A_i is absolutely preferred to A_j, then $a_{ij} = 9$, in the opposite case $a_{ij} = 1/9$, while indifference is indicated by $a_{ij} = 1$. Intermediate values correspond to intermediate judgements.

A multiplicative *PCM* is called *consistent* if and only if

$$a_{ih}a_{hj} = a_{ij}\ , \quad i, j, h = 1, ...n. \tag{1}$$

If a multiplicative *PCM* $A = [a_{ij}]$ is consistent, then a positive vector $w = [w_i]$ exists such that $a_{ij} = w_i/w_j,\ \ i, j = 1, ..., n$.

In the second approach, also called *additive approach* [2], the expert's preferences are described by a fuzzy preference relation $r : \Lambda \times \Lambda \to [0, 1]$ [5] [21] [22]. In this framework, $r(A_i, A_j)$ (for conciseness denoted by r_{ij} in the following) indicates the preference degree of alternative A_i over alternative A_j. It follows that $r_{ij} = 0.5$ indicates indifference between A_i and A_j, $r_{ij} = 1$ indicates that A_i is definitely preferred over A_j, and $r_{ij} = 0$ indicates the opposite case. The $n \times n$ matrix $R = [r_{ij}]$ is called *additive PCM*. Matrix R is nonnegative and reciprocal in the additive sense: $r_{ji} = 1 - r_{ij},\ \ i, j = 1, ..., n$.

An additive *PCM* $R = [r_{ij}]$ is called *consistent* (in additive sense) if and only if

$$(r_{ih} - 0.5) + (r_{hj} - 0.5) = (r_{ij} - 0.5) \qquad i, j, h = 1, ..., n. \tag{2}$$

Property (2) can also be written as

$$r_{ih} + r_{hj} - r_{ij} - 0.5 = 0 \qquad i, j, h = 1, ..., n. \tag{3}$$

By writing (2), the role of the differences from indifference value 0.5 is emphasized.

If an additive *PCM* $R = [r_{ij}]$ is consistent, then a positive vector $u = [u_i]$ of utility values exists such that

$$r_{ij} = 0.5 + 0.5(u_i - u_j) \quad i, j = 1, ..., n \tag{4}$$

with $| u_i - u_j | \leq 1 \ \forall ij$ [21].

In this paper we assume that the preferences are expressed following this additive approach. Nevertheless, it can be shown that the two approaches are equivalent: a simple function introduced in [6],

$$r_{ij} = f(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij}) \tag{5}$$

transforms the a_{ij} values into the r_{ij} values in such a way that all the relevant properties of $A = [a_{ij}]$ are transformed in the corresponding properties for $R = [r_{ij}]$ in the additive sense. In particular, multiplicative reciprocity is transformed in additive reciprocity and multiplicative consistency is transformed in additive consistency. This topic has been then developed in several other papers [2] [5] [12] [13] [14] [25]. A comprehensive and elegant survey on the relationships between the additive and the multiplicative approach (with slightly different definitions) can be found in [2].

2.2 Some known methods for incomplete comparisons

Let us now assume that one or more comparisons are missing. As a consequence, the *PCM* is incomplete and it is no longer possible to derive the priorities for the alternatives using the well known methods of the eigenvector, or the geometric mean, to cite the most popular ones. Some methods have been proposed in the literature to solve the incomplete comparison problem. Most of these methods are formulated in the multiplicative framework [4], [10], [11], [15], [19], some other in the additive framework [1], [23], [24]. An incomplete additive *PCM* is also referred to as incomplete fuzzy preference relation. Let us very briefly describe the most important (in our opinion) ideas presented in the above–mentioned literature.

Two methods have been proposed by P.T. Harker. The first one [11], called the *geometric mean method*, is based on the concept of "connecting path". If alternatives A_i and A_j are not compared with each other, let us denote by $\{A_i, A_j\}$ the missing comparison (*MC* in the following) and let x_{ij} be the corresponding numerical value to be estimated; a connecting path of size r has the following form

$$x_{ij} = a_{ik_1} a_{k_1 k_2} \cdots a_{k_r j} \tag{6}$$

where the comparison values at the r.h.s. are known. The connecting path of size two, also called *elementary connecting path*, corresponds to the more familiar expression (see (1))

$$x_{ij} = a_{ik}a_{kj}. \tag{7}$$

Note that each connecting path corresponds to an indirect comparison between A_i and A_j. Harker proposes that the value $\bar{x}_{ij}$ of the *MC* should be equal to the geometric mean of all connecting paths related to this *MC*. The drawback of this method is that the number of connecting paths grows with the number n of the alternatives in such a way that the method becomes computationally intractable for many real world problems.

The second Harker's method [10] is based on the following idea. The missing (i, j)–component is set to be equal to w_i/w_j, where the components of the vector w are not known and are to be calculated. In other words, the missing entries are completed by setting them equal to the value they should approximate. The matrix obtained with the described substitution is called C. An auxiliary nonnegative numerical matrix A is then associated to C satisfying $Aw = Cw$. The matrix A is nonnegative and *quasi reciprocal*, in the sense that all its positive entries are reciprocal, but it contains entries equal to zero. In this way, Harker transforms the original problem in that of computing the principal eigenvector of a nonnegative quasi reciprocal matrix. In order to justify his method, Harker develops a theory for such type of matrices, following the Saaty's one for positive reciprocal matrices.

Shiraishi et al. in [19] propose a heuristic method which is based on a property of a coefficient of the characteristic polynomial of a *PCM* A. More precisely, the coefficient c_3 of λ^{n-3} is viewed as an index of consistency for A. Therefore, in order to maximize the consistency of the *PCM*, the authors consider the missing entries in the *PCM* as variables $x_1, \ldots, x_m$ and propose to maximize $c_3(x_1, \ldots, x_m)$ as a function of these variables.

In [4] a Least Squares type method is proposed. Instead of first calculating the missing entries of a *PCM*, the priority vector $w = [w_i]$ is directly calculated as the solution of a constrained optimization problem. Here the variables are the n components w_i of w and only the known entries a_{ij} are approximated by w_i/w_j. The corresponding error is minimized as a function of $w_1, \ldots, w_n$.

In [23], Xu proposes to calculate the priority vector $w = [w_i]$ of incomplete fuzzy preference relation by a goal programming approach. His method refers to (4) and minimizes the errors $\varepsilon_{ij} = |r_{ij} - 0.5 + 0.5(u_i - u_j)|$ for all missing entries (i, j). He also proposes his goal programming approach with another type of consistency, different from (4).

In his second proposal, Xu [24] develops a method, for incomplete fuzzy preference relations, similar to that introduced by Harker [10] for incomplete multiplicative *PCM*. In [24] the priority vector $w = [w_i]$ is calculated by solving a system of equations which corresponds to the Harker's auxiliary eigenproblem.

In [1] an iterative method is proposed to evaluate the *MCs* in an incomplete fuzzy preference relations. The main idea is the following. If r_{ij} is unknown, the corresponding consistent value is calculated for each known indirect comparison between A_i and A_j (i.e. for each *elementary connecting path*); the arithmetic mean of all these values is the estimated value for r_{ij} (the approach is therefore in the spirit of [11]). The estimated values for the *MCs* are then iteratively utilized as known entries.

3 A new method for incomplete comparisons

Our method is based on an (in)consistency index introduced in [7], which directly refers to the definition (3) of consistency for a *PCM*.

As mentioned before, we assume that the preferences are expressed by an additive *PCM* $R = [r_{ij}], \;\; r_{ij} \in [0, 1]$. Following [7] and taking into account (3), let

$$L_{ijh} = (r_{ih} + r_{hj} - r_{ij} - 0.5)^2 \tag{8}$$

be the inconsistency contribution associated with the triplet of alternatives $\{A_i, A_j, A_h\}$. This definition is meaningful, as the following lemma holds.

Lemma 1

L_{ijh} is invariant under permutations of the indices.

Proof

The proof is based on the additive reciprocity of $R: \; r_{ji} = 1 - r_{ij}$. Let us rewrite L_{ijh} in the form $L_{ijh} = (r_{ih} + r_{hj} + r_{ji} - 1.5)^2$. It follows that

$$\begin{aligned}
L_{hij} &= (r_{hj} + r_{ji} + r_{ih} - 1.5)^2 = L_{ijh} \\
L_{jhi} &= (r_{ji} + r_{ih} + r_{hj} - 1.5)^2 = L_{ijh} \\
L_{ihj} &= (r_{ij} + r_{jh} + r_{hi} - 1.5)^2 = (1.5 - r_{ji} - r_{hj} - r_{ih})^2 = L_{ijh} \\
L_{hji} &= (r_{hi} + r_{ij} + r_{jh} - 1.5)^2 = (1.5 - r_{ih} - r_{ji} - r_{hj})^2 = L_{ijh} \\
L_{jih} &= (r_{jh} + r_{hi} + r_{ij} - 1.5)^2 = (1.5 - r_{hj} - r_{ih} - r_{ji})^2 = L_{ijh}
\end{aligned}$$

□

Two other useful properties of L_{ijh} are very easy to check:

- If $i \neq j \neq h$, then $L_{ijh} = 0$ iff the alternatives A_i, A_j, A_h are compared in a perfectly consistent way.
- If at least two indices are equal, then

$$L_{ijh} = 0. \tag{9}$$

As L_{ijh} corresponds to a local inconsistency contribution, the global inconsistency index ρ of matrix R is defined as (see [7])

$$\rho = \sum_{i,j,h=1}^{n} L_{ijh}. \tag{10}$$

Taking into account (9), ρ can be expressed as $\rho = \sum_{i \neq j \neq h} L_{ijh}$. Moreover, it is possible to consider in (10) the sum of only the $\binom{n}{3}$ terms corresponding to all the (non ordered) triplets of alternatives, thus avoiding repetitions; from lemma 1 we have: $\rho = 6 \sum_{i<j<h} L_{ijh}$. As we are interested only in optimization of ρ, the numerical factor 6 is irrelevant. In the following, we will use the simpler expression (10).

The method we propose for the incomplete comparison problem is based on the idea to consider the missing entries in the incomplete PCM as variables and calculate them by minimizing the global inconsistency index ρ; in this way, the values obtained are those that are most consistent with the available data.

3.1 The single missing comparison case

Let us assume that only one comparison is missing: $\{A_s, A_t\}$, i.e. the comparison between A_s and A_t, $s \neq t$. Then the two elements r_{st} and $r_{ts} = 1 - r_{st}$ are unknown in the $n \times n$ matrix R. The optimal value r^*_{st} is the solution of the following problem:

$$\min_{r_{st} \in [0,1]} \rho(r_{st}). \tag{11}$$

In order to minimize $\rho = \rho(r_{st})$, let us calculate the stationary point(s) of ρ. Taking into account Lemma 1, we have (in the following all the sums are made with respect to the index h)

$$
\begin{aligned}
\frac{d\rho}{dr_{st}} &= 6\frac{d}{dr_{st}}\sum_{\substack{h \neq s \\ h \neq t}} L_{sth} \\
\frac{1}{6}\frac{d\rho}{dr_{st}} &= \sum_{\substack{h \neq s \\ h \neq t}} (2r_{st} - 2r_{sh} - 2r_{ht} + 1) \\
&= 2((n-2)r_{st} - \sum_{h\neq t} r_{sh} - \sum_{h\neq s} r_{ht} + n/2). \qquad (12)
\end{aligned}
$$

By setting the derivative equal to zero we obtain

$$
\hat{r}_{st} = \frac{1}{n-2}\left[\sum_{h\neq t} r_{sh} + \sum_{h\neq s} r_{ht} - n/2\right]. \qquad (13)
$$

Function $\rho(r_{st})$ is a sum of strictly convex functions L_{sth}, so it is itself a strictly convex function. As a consequence, (13) gives the global minimum of $\rho(r_{st})$. The optimal value $\hat{r}_{st}$ given by (13) can be outside the interval $[0, 1]$, as it is well known that consistency may be incompatible with the use of a bounded scale; for example, given r_{ih}, $r_{hj} \in [0, 1]$, it is possible that no $r_{ij} \in [0, 1]$ exists such that (3) holds. As an example, consider the extreme values $r_{ih} = r_{hj} = 0$ or $r_{ih} = r_{hj} = 1$. Nevertheless, in this one–dimensional convex case, if $\hat{r}_{st} \notin [0, 1]$, then the (bounded) optimal solution r^*_{st} of (11) is simply obtained by taking $r^*_{st} = 0$ if $\hat{r}_{st} < 0$ and $r^*_{st} = 1$ if $\hat{r}_{st} > 1$.

3.2 The two missing comparisons case

As the general case of p missing comparisons needs a rather complex notation, let us first assume that only two comparisons are missing, say $\{A_s, A_t\}$ and $\{A_u, A_v\}$. If the two *MCs* do not share any alternative, i.e. indices s, t, u, v are all different each other, the optimal values of r_{ts} and r_{uv} can be calculated independently as described in the previous section, i.e. using (13).

Let us then assume that an alternative, say A_t, is involved in both *MCs*, which become $\{A_s, A_t\}$ and $\{A_u, A_t\}$. The four unknown elements in the $n \times n$ matrix R are r_{st} , $r_{ts} = 1 - r_{st}$, r_{ut} and $r_{tu} = 1 - r_{ut}$. The optimal values r^*_{st} and r^*_{ut} are computed by solving the following problem:

$$
\min_{(r_{st}, r_{ut})\in[0,1]^2} \rho(r_{st}, r_{ut}). \qquad (14)
$$

In order to minimize $\rho = \rho(r_{st}, r_{ut})$, let us first calculate the stationary point(s) of $\rho(r_{st}, r_{ut})$

$$
\begin{aligned}
\frac{\partial\rho}{\partial r_{st}} &= 6\frac{\partial}{\partial r_{st}} \sum_{h\neq s, h\neq t} L_{sth} \\
\frac{1}{6}\frac{\partial\rho}{\partial r_{st}} &= \sum_{h\neq s, h\neq t} \frac{\partial L_{sth}}{\partial r_{st}} \\
&= \sum_{h\neq s, h\neq t} (2r_{st} - 2r_{sh} - 2r_{ht} + 1) \\
&= 2(n-2)r_{st} - 2\sum_{h\neq s, h\neq t} r_{sh} - 2\sum_{h\neq s, h\neq u, h\neq t} r_{ht} - 2r_{ut} + (n-2).
\end{aligned}
$$

By setting $\partial\rho/\partial r_{st} = 0$ we obtain

$$
\begin{aligned}
2(n-2)r_{st} - 2r_{ut} &= 2\sum_{h\neq s, h\neq t} r_{sh} + 2\sum_{h\neq s, h\neq u, h\neq t} r_{ht} - n + 2 \\
(n-2)r_{st} - r_{ut} &= \sum_{h\neq s, h\neq t} r_{sh} + \sum_{h\neq s, h\neq u, h\neq t} r_{ht} - n/2 + 1 \\
(n-2)r_{st} - r_{ut} &= \sum_{h\neq t} r_{sh} + \sum_{h\neq s, h\neq u} r_{ht} - n/2.
\end{aligned}
$$

Analogously, we obtain

$$
\frac{1}{6}\frac{\partial\rho}{\partial r_{ut}} = 2(n-2)r_{ut} - 2\sum_{h\neq u, h\neq t} r_{uh} - 2\sum_{h\neq s, h\neq u, h\neq t} r_{ht} - 2r_{st} + (n-2)
$$

and, by setting $\partial\rho/\partial r_{ut} = 0$,

$$
(n-2)r_{ut} - r_{st} = \sum_{h\neq t} r_{uh} + \sum_{h\neq s, h\neq u} r_{ht} - n/2.
$$

By imposing both derivatives equal to zero, we obtain the system

$$
\left\{
\begin{aligned}
(n-2)r_{st} - r_{ut} &= \textstyle\sum_{h\neq t} r_{sh} + \sum_{h\neq s, h\neq u} r_{ht} - n/2 \\
-r_{st} + (n-2)r_{ut} &= \textstyle\sum_{h\neq t} r_{uh} + \sum_{h\neq s, h\neq u} r_{ht} - n/2
\end{aligned}
\right. \tag{15}
$$

where the coefficient matrix is

$$Q = \begin{bmatrix} n-2 & -1 \\ -1 & n-2 \end{bmatrix} \tag{16}$$

and the r.h.s. vector is

$$b = \begin{bmatrix} \sum_{h\neq t} r_{sh} + \sum_{h\neq s, h\neq u} r_{ht} - n/2 \\ \sum_{h\neq t} r_{uh} + \sum_{h\neq s, h\neq u} r_{ht} - n/2 \end{bmatrix}. \tag{17}$$

The coefficient matrix is clearly nonsingular and therefore system (15) has a unique solution $(\hat{r}_{st}, \hat{r}_{ut}) = Q^{-1}b$ which is the single stationary point of the function $\rho(r_{st}, r_{ut})$. The function $\rho(r_{st}, r_{ut})$ is convex, as it is a sum of convex functions, so that the stationary point is a global minimum point. The convexity of $\rho(r_{st}, r_{ut})$ can be also directly checked by its (constant) hessian matrix

$$H = 6 \begin{bmatrix} 2(n-2) & -2 \\ -2 & 2(n-2) \end{bmatrix}, \tag{18}$$

which is positive definite for $n > 3$. If $(\hat{r}_{st}, \hat{r}_{ut})$ belongs to $[0, 1]^2$, it is the solution of problem (14). As in the one–missing comparison case, when the available judgements are seriously inconsistent, it may happen that $(\hat{r}_{st}, \hat{r}_{ut}) \notin [0, 1]^2$. In this case the constrained problem should be solved by means of a quadratic programming algorithm [9].

3.3 The p missing comparisons case

As in the previous cases, let us associate a variable r_{st} to each *MC* $\{A_s, A_t\}$. If p comparisons are missing, then the matrix R contains $2p$ unknown elements, due to its reciprocity. In this general case, analogously to the two–dimensional one, it is necessary to distinguish between two different type of *MCs*.

The first one, which we call *independent*, occurs when in a *MC* $\{A_s, A_t\}$ neither of the alternatives A_s and A_t is involved in any other *MC*. That is: $\{A_s, A_t\} \cap \{A_i, A_j\} = \emptyset$ for any other *MC* $\{A_i, A_j\}$. In this case the optimal value $\hat{r}_{st}$ of the variable r_{st} is simply given by (13).

We now focus on sets of *dependent MCs*; we call *dependent* the *MCs* of a set such that for every partition into two subsets, there exists at least one alternative which is in both subsets. As an example, $\{\{A_1, A_2\}, \{A_2, A_4\}, \{A_4, A_7\}\}$ is a dependent set of *MCs*, while $\{\{A_1, A_2\}, \{A_2, A_4\}, \{A_5, A_7\},$

$\{A_5, A_8\}\}$ is not, as the two subsets $\{\{A_1, A_2\}, \{A_2, A_4\}\}$ and $\{\{A_5, A_7\}, \{A_5, A_8\}\}$ do not contain a common alternative. As a consequence of these definitions, the p *MCs* can be divided in a certain number of independent *MCs* and some disjoint sets of dependent *MCs*.

As mentioned before, it is easy to find the optimal value of a variable r_{st} associated with an *independent MC*, as it can be calculated *independently* from the other variables. Analogously, we can calculate the optimal value of the variables associated with a set of *dependent MCs independently* from all the other variables not referring to that set.

In the following, we describe how to calculate the optimal values of a set of $m \leq n-2$ variables associated with a set of *dependent MCs*. In order to simplify the indices notation, we can ignore all the other $p-m$ *MCs* not in the dependent set, assuming that only those m comparisons are missing in the problem. We denote the m pair of alternatives corresponding to the m *MCs* by $\{A_{s_k}, A_{t_k}\}, k = 1, ..., m$. Then ρ is a function of the m variables $x_k = r_{s_k t_k},\ k = 1, ..., m$. Without loss of generality, we assume, that $s_k < t_k$. The matrix R has $2m$ unknown elements: the m variables $r_{s_k t_k},\ k = 1, ..., m$ above the diagonal and the m corresponding reciprocal elements below the diagonal. For each variable $r_{s_k t_k}$, let us denote with $IR^+_{s_k}$ and $IR^-_{s_k}$ the sets of column indices of the other unknown elements which are on the same row s_k respectively above and below the diagonal. Analogously, let us denote with $IC^+_{t_k}$ and $IC^-_{t_k}$ the sets of row indices of the other unknown elements which are on the same column t_k respectively above and below the diagonal. The example in the next subsection will clarify this notation.

3.3.1 Consistency optimization

Analogously to the one and two–dimensional cases, in order to solve the problem

$$\min_{(r_{s_1 t_1}, ..., r_{s_m t_m}) \in [0,1]^m} \rho(r_{s_1 t_1}, ..., r_{s_m t_m}), \tag{19}$$

let us calculate its m partial derivatives and set them equal to zero. From (12) and (13) we have

$$\frac{\partial \rho}{\partial r_{s_k, t_k}} = 0$$

$$(n-2) r_{s_k, t_k} - \sum_{h \neq t_k} r_{s_k h} - \sum_{h \neq s_k} r_{h t_k} + n/2 = 0$$

and, after some calculation,

$$(n-2)r_{s_k,t_k} - \sum_{h\in IR^+_{s_k}} r_{s_k h} + \sum_{h\in IR^-_{s_k}} r_{h s_k} - \sum_{h\in IC^+_{t_k}} r_{h t_k} + \sum_{h\in IC^-_{t_k}} r_{t_k h} \quad =$$

$$Card(IR^-_{s_k}) + Card(IC^-_{t_k}) + \sum_{h\notin(IR^+_{s_k}\cup IR^-_{s_k})} r_{s_k h} + \sum_{h\notin(IC^+_{t_k}\cup IC^-_{t_k})} r_{h t_k} - n/2. \quad (20)$$

Taking (20) for $k = 1, ..., m$, we impose the m derivatives $\partial\rho/\partial r_{s_k,t_k}$ equal to zero, thus obtaining a linear system with m equations in the m variables $r_{s_k t_k},\ k = 1, ..., m$:

$$Qx = b, \quad (21)$$

where

$$Q = \begin{bmatrix} n-2 & q_{1,2} & q_{1,3} & ... & ... & q_{1,m} \\ q_{2,1} & n-2 & q_{2,3} & ... & ... & q_{2,m} \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ q_{m-1,1} & q_{m-1,2} & ... & ... & n-2 & q_{m-1,m} \\ q_{m,1} & q_{m,2} & ... & ... & q_{m,m-1} & n-2 \end{bmatrix} \quad (22)$$

$$x = \begin{bmatrix} r_{s_1 t_1} \\ . \\ . \\ r_{s_k t_k} \\ . \\ . \\ r_{s_m t_m} \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ . \\ . \\ b_k \\ . \\ . \\ b_m \end{bmatrix},$$

$$b_k = card(IR^-_{s_k}) + card(IC^-_{t_k}) + \sum_{h\notin(IR^+_{s_k}\cup IR^-_{s_k})} r_{s_k h} + \sum_{h\notin(IC^+_{t_k}\cup IC^-_{t_k})} r_{h t_k} - n/2$$

$$q_{k,h} = \begin{cases} n-2 & \text{if} \quad h = k \\ -1 & \text{if} \quad t_h \in IR^+_{s_k} \quad \text{or} \quad s_h \in IC^+_{t_k} \\ +1 & \text{if} \quad t_h \in IR^-_{s_k} \quad \text{or} \quad s_h \in IC^-_{t_k} \\ 0 & \text{otherwise} \end{cases}$$

or, more simply:

$$q_{k,h} = \begin{cases} n-2 & \text{if} \quad h = k \\ -1 & \text{if} \quad s_h = s_k \quad \text{or} \quad t_h = t_k \\ +1 & \text{if} \quad s_h = t_k \quad \text{or} \quad t_h = s_k \\ 0 & \text{otherwise.} \end{cases}$$

Note that all information on the known comparisons are summarized in vector b, while matrix $Q = [q_{ij}]$ encodes the interdependence structure of the m MCs, with $q_{ij} \in \{-1, 0, 1\}$ for $i \neq j$ and $q_{ii} = n - 2\ \forall i$.

The following theorem on the optimal values of the missing comparisons is the main result of the paper.

Theorem 1

If $m \leq n - 2$, then

(a) Matrix Q is nonsingular and therefore system (21) has a unique solution given by $\hat{x} = Q^{-1}b$.

(b) Solution $\hat{x}$ is the global minimum point for function $\rho(x)$.

Proof

(a) From the assumption $m \leq n - 2$, it follows that Q is strictly diagonally dominant [1], $\sum_{j=1 j\neq i}^{m} |q_{i,j}| < m \leq n - 2 = |q_{i,i}|$ for all i. A strictly diagonally dominant matrix is nonsingular (theorem of Levy – Desplanques) and therefore invertible. Then system (21) has a unique solution given by $\hat{x} = Q^{-1}b$, i.e. the unique stationary point of $\rho(x)$. The first part of the theorem is then proved.

(b) From the definition of q_{hk}, it follows that $q_{hk} = q_{kh}$. Then Q is a symmetric matrix, strictly diagonally dominant, with all the diagonal elements positive; this is sufficient to conclude that Q is positive definite. The hessian matrix of $\rho(x)$ is itself positive definite, being a positive multiple of Q. Then $\rho(x)$ is a strictly convex function and the stationary point $\hat{x}$ is its global minimum point. $\square$

As in the previous cases, when the available judgements are seriously inconsistent, it may happen that $\hat{x} \notin [0, 1]^m$. Also in this case the minimum point in $[0, 1]^m$ can be obtained by solving a quadratic programming problem.

The possibility of obtaining a numerical result not belonging to the a priori chosen feasible set of values is well known and unavoidable when dealing with inconsistency and a bounded scale. Clearly the problem does not exist when an open scale is used [2]. On the other hand, despite the elegant mathematical results, every unbounded scale yields serious drawbacks in practical applications, but we do not want to dwell deeper on this issue.

Example

[1] A square $m \times m$ matrix $A = [a_{ij}]$ is called diagonally dominant if $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|,\ i = 1, ..., m$. The matrix is called strictly diagonally dominant if strict inequality holds.

As the notation required by the general p–dimensional case described above is rather complex, we try to clarify the preceding issues by the following example. Let us consider a case with nine alternatives $\{A_1, A_2, ..., A_9\}$; we do not take into account the known entries in the 9×9 matrix and we focus on the following five *MCs* with their corresponding variables

missing comparison	variable	k
$\{A_1, A_5\}$	$r_{15} = x_1$	1
$\{A_1, A_9\}$	$r_{19} = x_2$	2
$\{A_3, A_5\}$	$r_{35} = x_3$	3
$\{A_5, A_7\}$	$r_{57} = x_4$	4
$\{A_5, A_9\}$	$r_{59} = x_5$	5

In the resulting *PCM* R we indicate only the unknown entries:

$$R = \begin{bmatrix} & & r_{15} & & r_{19} \\ & & r_{35} & & \\ 1-r_{15} & 1-r_{35} & & r_{57} & r_{59} \\ & & 1-r_{57} & & \\ 1-r_{19} & & 1-r_{59} & & \end{bmatrix}. \tag{23}$$

The l.h.s. part of system (21) becomes

$$\begin{cases} 7r_{15} & -r_{19} & -r_{35} & +r_{57} & +r_{59} & = & \cdots \\ -r_{15} & 7r_{19} & 0 & 0 & -r_{59} & = & \cdots \\ -r_{15} & 0 & 7r_{35} & +r_{57} & +r_{59} & = & \cdots \\ r_{15} & 0 & +r_{35} & +7r_{57} & -r_{59} & = & \cdots \\ r_{15} & -r_{19} & +r_{35} & -r_{57} & +7r_{59} & = & \cdots \end{cases} \tag{24}$$

Clearly, we cannot write the vector b on the r.h.s. of the system unless we assign all the known entries. The coefficient matrix of the system (24) is

$$Q = \begin{bmatrix} 7 & -1 & -1 & 1 & 1 \\ -1 & 7 & 0 & 0 & -1 \\ -1 & 0 & 7 & 1 & 1 \\ 1 & 0 & 1 & 7 & -1 \\ 1 & -1 & 1 & -1 & 7 \end{bmatrix} \tag{25}$$

which is symmetric, positive definite and therefore nonsingular. With the help of the known entries of *PCM* R (whatever they are) is then possible to calculate vector b, so that the solution of the system (24) is given by $\hat{x} = Q^{-1}b$.

4 Numerical results

We propose some numerical experiences on a problem, concerning the choice of a job, which has been proposed and studied by T. Saaty in [18], page 85. The pairwise comparison matrix obtained by Saaty is transformed, by means of (5), into the equivalent *additive* matrix

$$R = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.8155 & 0.5 & 0.3423 \\ 0.5 & 0.5 & 0.6577 & .81550 & 0.5 & 0.3423 \\ 0.5 & 0.3423 & 0.5 & 0.8662 & 0.75 & 0.3423 \\ 0.1845 & 0.1845 & 0.1338 & 0.5 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.25 \\ 0.6577 & 0.6577 & 0.6577 & 0.75 & 0.75 & 0.5 \end{bmatrix}. \tag{26}$$

The inconsistency index (10) of R is $\rho = 3.78$. Now, let us assume that the decision maker is not able to perform the comparison between alternatives A_2 and A_3. Then R is incomplete, as the two elements r_{23} and $r_{32} = 1 - r_{23}$ are unknown. We complete R using our method, i.e. by minimizing $\rho(r_{23})$, as explained in section 3.1. We obtain $r_{23} = 0.4248$ and the inconsistency index of R becomes $\rho = 2.478$. If, in addition to $\{A_2, A_3\}$, the comparison $\{A_3, A_5\}$ is missing as well, we analogously calculate the optimal values of r_{23} and r_{35} by solving (15). The obtained values are $r_{23} = 0.4726$ and $r_{35} = 0.559$ with the inconsistency decreased to $\rho = 1.6577$. Finally, if we consider as missing also a third comparison $\{A_4, A_6\}$, we have to solve the system (21) to calculate the optimal values of the three variables, obtaining: $r_{23} = 0.4726$, $r_{35} = 0.559$ and $r_{46} = 0.0074$. The matrix R, completed with these values, is very close to consistency, being $\rho = 0.245$. Note that in all the cases mentioned above the obtained values belong to $[0, 1]$.

5 Conclusions and final remarks

We think that our proposal is a natural way to solve the problem of missing data in pairwise comparison. We propose to complete the *PCM* coherently with the available judgements by directly referring to the definition (3) of consistency. From the computational point of view our method is simple: in order to calculate the optimal values, we only have to solve a nonsingular linear system. Our future research effort will be directed at comparing our method with other approaches by suitable numerical experiments in order to highlight different properties / characteristics.

By means of (10) we define the global inconsistency index ρ of a *PCM* R simply by summing all the local inconsistency contributions L_{ijh}. If we are interested in defining an index independent from the order n of the matrix, it is clearly possible to apply the necessary normalization. By taking into account only the non trivial elements in (10) and dividing by their number, a mean value of L_{ijh} is obtained, $\bar{L}_{ijh} = \sum_{i<j<h} L_{ijh}/\binom{n}{3}$, which is the suitable order independent inconsistency index.

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